Logic and Topology for Knowledge, Knowability, and Belief

Extended Abstract

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In recent work, Stalnaker proposes a logical framework in which belief is realized as a weakened form of knowledge [33]. Building on Stalnaker's core insights, and using frameworks developed in [12] and [4], we employ *topological* tools to refine and, we argue, improve on this analysis. The structure of topological subset spaces allows for a natural distinction between what is *known* and (roughly speaking) what is *knowable*; we argue that the foundational axioms of Stalnaker's system rely intuitively on *both* of these notions. More precisely, we argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between an "evidence-in-hand" conception of knowledge and a weaker "evidence-out-there" notion of what *could come to be known*. Our analysis leads to a trimodal logic of knowledge, knowability, and belief interpreted in topological subset spaces in which belief is definable in terms of *knowledge and knowability*. We provide a sound and complete axiomatization for this logic as well as its uni-modal belief fragment. We then consider weaker logics that preserve suitable translations of Stalnaker's postulates, yet do not allow for any reduction of belief. We propose novel topological semantics for these irreducible notions of belief, generalizing our previous semantics, and provide sound and complete axiomatizations for the corresponding logics.

1 Introduction

Epistemology has long been concerned with the relationship between knowledge and belief. There is a long tradition of attempting to strengthen the latter to attain a satisfactory notion of the former: belief might be improved to true belief, to "justified" true belief, to "correctly justified" true belief [13], to "undefeated justified" true belief [28, 29, 27, 26], and so on (see, e.g, [25, 32] for a survey). There has also been some interest in reversing this project—deriving belief from knowledge—or, at least, putting "knowledge first" [36]. In this spirit, Stalnaker has proposed a framework in which belief is realized as a weakened form of knowledge [33]. More precisely, beginning with a logical system in which both belief and knowledge are represented as primitives, Stalnaker formalizes some natural-seeming relationships between the two, and proves on the basis of these relationships that belief can be *defined* out of knowledge.

This project is of both conceptual and technical interest. Philosophically speaking, it provides a new perspective from which to investigate knowledge, belief, and their interplay. Mathematically, it offers a potential route by which to represent belief in formal systems that are designed to handle only knowledge. Both these themes underlie the present work. Building on Stalnaker's core insights, we employ *topological* tools to refine and, we argue, improve on Stalnaker's original system.

Our work brings together two distinct lines of research. Stalnaker's epistemic-doxastic axioms have motivated and inspired several prior topological proposals for the semantics of belief [31, 2, 5, 4], in-

cluding most recently and most notably a proposal by Baltag et al. [4] that is essentially recapitulated in our strongest logic for belief (Section 3). Our development of this logic, however (as well as the new, weaker logics we study in Section 4), relies crucially on a semantic framework defined in recent work by Bjorndahl [12] that distinguishes what is *known* from (roughly speaking) what is *knowable*.

We argue that the foundational axioms of Stalnaker's system rely intuitively on *both* of these notions at various points. More precisely, we argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between an "evidence-in-hand" conception of knowledge and a weaker "evidence-out-there" notion of what *could come to be known*. As such, we find it quite natural to study Stalnaker's principles in the richer semantic setting developed in [12], which is based on *topological subset spaces*, a class of epistemic models of growing interest in recent years [30, 14, 12, 16, 17]. These models support a careful reworking of Stalnaker's system in a manner that respects the distinction described above, yielding a trimodal logic of knowledge, knowability, and belief that is our main object of study.

Subset spaces have been employed in the representation of a variety of epistemic notions, including knowledge, learning, and public announcement (see, e.g., [30, 22, 9, 8, 1, 35, 34, 23]), but to the best of our knowledge this paper contains the first formalization of *belief* in subset space semantics. Stalnaker's original system is an extension of the basic logic of knowledge S4; belief emerges as a standard KD45 modality, as it is often assumed to be, while knowledge turns out to satisfy the stronger but somewhat less common S4.2 axioms. Our system, by contrast, is an extension of the basic *bimodal* logic of knowledge-and-knowability introduced in [12]; belief is similarly KD45, while knowledge is S5 and knowability is S4; thus, our approach preserves what are arguably the desirable properties of belief while cleanly dividing "knowledge" into two conceptually distinct and familiar logical constructs.

In Stalnaker's system, belief can be defined in terms of knowledge; in our system, we prove that belief can be defined in terms of *knowledge and knowability* (Proposition 3). This yields a purely topological interpretation of belief that coincides with that previously proposed by Baltag et al. [4]: roughly speaking, while knowledge is interpreted (as usual) as "truth in all possible alternatives", belief becomes "truth in *most* possible alternatives", with the meaning of "most" cashed out topologically. The conceptual underpinning of this interpretation of belief as developed by Baltag et al., and its connection to the present work, is discussed further in Section 5.

In this richer topological setting, the translation of Stalnaker's postulates do not in themselves entail that belief is reducible to knowledge (or even knowledge-and-knowability): our characterization of belief in these terms relies on two additional principles we call "weak factivity" and "confident belief". This motivates the study of weaker logical systems obtained by rejecting one or both of these principles. We initiate the investigation of these systems by proposing novel topological semantics that aim to capture the corresponding, irreducible notions of belief.

This rest of the paper is organized as follows. In Section 2 we present Stalnaker's original system, motivate our objections to it, and introduce the formal logical framework that supports our revision. In Section 3 we present our revised system, explore its relationship to Stalnaker's system, and prove an analogue to Stalnaker's characterization result: belief can be defined out of knowledge *and knowability*. We also establish that our system is sound and complete with respect to the class of topological subset models, and that the pure logic of belief it embeds is axiomatized by the standard KD45 system. In Section 4 we investigate weaker logics as discussed above and develop the semantic tools needed to interpret belief in this more general context; we also provide soundness and completeness results for each of these logics. Section 5 concludes. Due to length restrictions, several of the longer proofs are omitted from the main body; they can all be found in the full version of this paper at `https://arxiv.org/abs/1612.02055`.

2 Knowledge, Knowability, and Belief

Given unary modalities $\star_1, \ldots, \star_k$, let $\mathcal{L}_{\star_1, \ldots, \star_k}$ denote the propositional language recursively generated by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \star_i \varphi,$$

where $p \in \text{PROP}$, the (countable) set of *primitive propositions*, and $1 \le i \le k$. Our focus in this paper is the trimodal language $\mathcal{L}_{K, \Box, B}$ and various fragments thereof, where we read $K\varphi$ as "the agent knows φ", $\Box\varphi$ as "φ is knowable" or "the agent could come to know φ", and $B\varphi$ as "the agent believes φ". The Boolean connectives $\vee$, $\to$ and $\leftrightarrow$ are defined as usual, and $\bot$ is defined as an abbreviation for $p \wedge \neg p$. We also employ $\hat{K}$ as an abbreviation for $\neg K\neg$, $\Diamond$ for $\neg\Box\neg$, and $\hat{B}$ for $\neg B\neg$.

(K$_\star$)	$\vdash \star(\varphi \to \psi) \to (\star\varphi \to \star\psi)$	Distribution
(D$_\star$)	$\vdash \star\varphi \to \neg\star\neg\varphi$	Consistency
(T$_\star$)	$\vdash \star\varphi \to \varphi$	Factivity
(4$_\star$)	$\vdash \star\varphi \to \star\star\varphi$	Positive introspection
(.2$_\star$)	$\vdash \neg\star\neg\star\varphi \to \star\neg\star\neg\varphi$	Directedness
(5$_\star$)	$\vdash \neg\star\varphi \to \star\neg\star\varphi$	Negative introspection
(Nec$_\star$)	from $\vdash \varphi$ infer $\vdash \star\varphi$	Necessitation

Table 1: Some axiom schemes and a rule of inference for $\star$

Let CPL denote an axiomatization of classical propositional logic. Then, following standard naming conventions, we define the following logical systems:

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\begin{aligned}
\mathsf{K}_\star &= \mathsf{CPL} + (\mathsf{K}_\star) + (\mathsf{Nec}_\star) \\
\mathsf{S4}_\star &= \mathsf{K}_\star + (\mathsf{T}_\star) + (4_\star) \\
\mathsf{S4.2}_\star &= \mathsf{S4}_\star + (.2_\star) \\
\mathsf{S5}_\star &= \mathsf{S4}_\star + (5_\star) \\
\mathsf{KD45}_\star &= \mathsf{K}_\star + (\mathsf{D}_\star) + (4_\star) + (5_\star).
\end{aligned}
$$

2.1 Stalnaker's system

Stalnaker [33] works with the language $\mathcal{L}_{K,B}$, augmenting the logic $\mathsf{S4}_K$ with the additional axioms schemes presented in Table 2. Let Stal denote this combined logic. Stalnaker proves that this system

(D$_B$)	$\vdash B\varphi \to \neg B\neg\varphi$	Consistency of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\vdash \neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$\vdash B\varphi \to BK\varphi$	Full belief

Table 2: Stalnaker's additional axiom schemes

yields the pure belief logic $\mathsf{KD45}_B$; moreover, he shows that Stal proves the following equivalence:

$B\varphi \leftrightarrow \hat{K}K\varphi$. Thus, belief in this system is reducible to knowledge; every formula of $\mathcal{L}_{K,B}$ can be translated into a provably equivalent formula in $\mathcal{L}_K$. Stalnaker also shows that although only the S4$_K$ system is assumed for knowledge, Stal actually derives the stronger system S4.2$_K$.

What justifies the assumption of these particular properties of knowledge and belief? It is, of course, possible to object to any of them (including the features of knowledge picked out by the system S4$_K$); however, in this paper we focus on the relationships expressed in (KB) and (FB). That knowing implies believing is widely taken for granted—loosely speaking, it corresponds to a conception of knowledge as a special kind of belief. Full belief,[1] on the other hand, may seem more contentious; this is because it is keyed to a rather strong notion of belief. The English verb "to believe" has a variety of uses that vary quite a bit in the nature of the attitude ascribed to the subject. For example, the sentence, "I believe Mary is in her office, but I'm not sure" makes a clearly possibilistic claim, whereas, "I believe that nothing can travel faster than the speed of light" might naturally be interpreted as expressing a kind of certainty. It is this latter sense of belief that Stalnaker seeks to capture: belief as *subjective certainty*. On this reading, (FB) essentially stipulates that being certain is not subjectively distinguishable from knowing: an agent who feels certain that φ is true also feels certain that she *knows* that φ is true.

At a high level, then, each of (KB) and (FB) have a certain plausibility. Crucially, however, we contend that their *joint* plausibility is predicated on an abstract conception of knowledge that permits a kind of equivocation. In particular, tension between the two emerges when knowledge is interpreted more concretely in terms of what is justified by a body of evidence.

Consider the following informal account of knowledge: an agent *knows* something just in case it is entailed by the available evidence. To be sure, this is still vague since we have not yet specified what "evidence" is or what "available" means (we return to formalize these notions in Section 2.2). But it is motivated by a very commonsense interpretation of knowledge, as for example in a card game when one player is said to *know* their opponent is not holding two aces on the basis of the fact that they are themselves holding three aces.

Even at this informal level, one can see that something like this conception of knowledge lies at the root of the standard *possible worlds semantics* for epistemic logic. Roughly speaking, such semantics work as follows: each world w is associated with a set of *accessible* worlds $R(w)$, and the agent is said to *know* φ *at* w just in case φ is true at all worlds in $R(w)$. A standard intuition for this interpretation of knowledge is given in terms of evidence: the worlds in $R(w)$ are exactly those compatible with the agent's evidence at w, and so the agent knows φ just in case the evidence rules out all not-φ possibilities. Suppose, for instance, that you have measured your height and obtained a reading of 5 feet and 10 inches ± 1 inch. With this measurement in hand, you can be said to *know* that you are less than 6 feet tall, having ruled out the possibility that you are taller.

Call this the *evidence-in-hand* conception of knowledge. Observe that it fits well with the (KB) principle: evidence-in-hand that entails φ should surely also cause you to believe φ. On the other hand, it does not sit comfortably with (FB): presumably you can be (subjectively) certain of φ without simultaneously being certain that you currently have evidence-in-hand that guarantees φ, lest we lose the distinction between belief and knowledge.[2] However, the intuition for (FB) can be recovered by shifting the meaning of "available evidence" to a weaker existential claim: that *there is* evidence entailing φ— even if you don't happen to personally have it in hand at the moment. This corresponds to a transition from the known to the knowable. On this account, (FB) is recast as "If you are certain of φ, then you are

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[1]Stalnaker calls this property "strong belief" but we, following [2, 3], adopt the term "full belief" instead.

[2]This assumes, roughly speaking, that evidence-in-hand is "transparent" in the sense that the agent cannot be mistaken about what evidence she has or what it entails. A model rich enough to represent this kind of uncertainty about evidence might therefore be of interest; we leave the development of such a model to other work.

certain that there is evidence entailing φ", a sort of dictum of responsible belief: do not believe anything unless you think you could come to know it. Returning to (KB), on the other hand, we see that it is not supported by this weaker sense of evidence-availability: the fact that you could, in principle, discover evidence entailing φ should not in itself imply that you believe φ.

This way of reconciling Stalnaker's proposed axioms with an evidence-based account of knowledge—namely, by carefully distinguishing between knowledge and knowability—is the focus of the remainder of this paper. We begin by defining a class of models rich enough to interpret both of these modalities at once.

2.2   Topological subset models

A **subset space** is a pair $(X, \mathcal{S})$ where X is a nonempty set of *worlds* and $\mathcal{S} \subseteq 2^X$ is a collection of subsets of X. A **subset model** $\mathcal{X} = (X, \mathcal{S}, v)$ is a subset space $(X, \mathcal{S})$ together with a function $v : \text{PROP} \to 2^X$ specifying, for each primitive proposition $p \in \text{PROP}$, its *extension* $v(p)$.

Subset space semantics interpret formulas not at worlds x but at *epistemic scenarios* of the form (x, U), where $x \in U \in \mathcal{S}$. Let $ES(\mathcal{X})$ denote the collection of all such pairs in $\mathcal{X}$. Given an epistemic scenario $(x, U) \in ES(\mathcal{X})$, the set U is called its *epistemic range*; intuitively, it represents the agent's current information as determined, for example, by the measurements she has taken. We interpret $\mathcal{L}_K$ in $\mathcal{X}$ as follows:

$$\begin{aligned}
(\mathcal{X}, x, U) &\models p && \text{iff} && x \in v(p) \\
(\mathcal{X}, x, U) &\models \neg\varphi && \text{iff} && (\mathcal{X}, x, U) \not\models \varphi \\
(\mathcal{X}, x, U) &\models \varphi \wedge \psi && \text{iff} && (\mathcal{X}, x, U) \models \varphi \text{ and } (\mathcal{X}, x, U) \models \psi \\
(\mathcal{X}, x, U) &\models K\varphi && \text{iff} && (\forall y \in U)((\mathcal{X}, y, U) \models \varphi).
\end{aligned}$$

Thus, knowledge is cashed out as truth in all epistemically possible worlds, analogously to the standard semantics for knowledge in relational models. A formula φ is said to be **satisfiable in** $\mathcal{X}$ if there is some $(x, U) \in ES(\mathcal{X})$ such that $(\mathcal{X}, x, U) \models \varphi$, and **valid in** $\mathcal{X}$ if for all $(x, U) \in ES(\mathcal{X})$ we have $(\mathcal{X}, x, U) \models \varphi$. The set

$$[\![\varphi]\!]_{\mathcal{X}}^U = \{x \in U : (\mathcal{X}, x, U) \models \varphi\}$$

is called the **extension of** φ **under** U. We sometimes drop mention of the subset model $\mathcal{X}$ when it is clear from context.

Subset space models are well-equipped to give an account of evidence-based knowledge and its *dynamics*. Elements of $\mathcal{S}$ can be thought of as potential pieces of evidence, while the epistemic range U of an epistemic scenario (x, U) corresponds to the "evidence-in-hand" by means of which the agent's knowledge is evaluated. This is made precise in the semantic clause for $K\varphi$, which stipulates that the agent knows φ just in case φ is entailed by the evidence-in-hand.

In this framework, stronger evidence corresponds to a smaller epistemic range, and whether a given proposition can come to be known corresponds (roughly speaking) to whether there exists a sufficiently strong piece of (true) evidence that entails it. This notion is naturally and succinctly formalized *topologically*.

A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T} \subseteq 2^X$ is a collection of subsets of X that covers X and is closed under finite intersections and arbitrary unions. The collection $\mathcal{T}$ is called a *topology on X* and elements of $\mathcal{T}$ are called *open* sets. In what follows we assume familiarity with basic topological notions; for a general introduction to topology we refer the reader to [19, 20].

A **topological subset model** is a subset model $\mathcal{X} = (X, \mathcal{T}, v)$ in which $\mathcal{T}$ is a topology on X. Clearly every topological space is a subset space. But topological spaces possess additional structure that enables

us to study the kinds of epistemic dynamics we are interested in. More precisely, we can capture a notion of knowability via the following definition: for $A \subseteq X$, say that x lies in the **interior** of A if there is some $U \in \mathcal{T}$ such that $x \in U \subseteq A$. The set of all points in the interior of A is denoted $int(A)$; it is easy to see that $int(A)$ is the largest open set contained in A. Given an epistemic scenario (x, U) and a primitive proposition p, we have $x \in int(\llbracket p \rrbracket^U)$ precisely when there is some evidence $V \in \mathcal{T}$ that is true at x and that entails p. We therefore interpret the extended language $\mathcal{L}_{K,\square}$ that includes the "knowable" modality in $\mathcal{X}$ via the additional recursive clause

$$(\mathcal{X}, x, U) \models \square \varphi \quad \text{iff} \quad x \in int(\llbracket \varphi \rrbracket^U).$$

The formula $\square \varphi$ thus represents knowability as a restricted existential claim over the set $\mathcal{T}$ of available pieces of evidence. The dual modality correspondingly satisfies

$$(\mathcal{X}, x, U) \models \Diamond \varphi \quad \text{iff} \quad x \in cl(\llbracket \varphi \rrbracket^U),$$

where cl denotes the topological closure operator.[3] Since the formula $\square \neg \varphi$ reads as "the agent could come to know that φ is false", one intuitive reading of its negation, $\Diamond \varphi$, is "φ is unfalsifiable".

It is worth noting that the intuition behind reading $\square \varphi$ as "φ is knowable" can falter when φ is itself an epistemic formula. In particular, if φ is the Moore sentence $p \wedge \neg K p$, then $K \varphi$ is not satisfiable in any subset model, so in this sense φ can never be known; nonetheless, $\square \varphi$ *is* satisfiable. Loosely speaking, this is because our language abstracts away from the implicit temporal dimension of knowability; $\square \varphi$ might be more accurately glossed as "one could come to know what φ *used to express* (before you came to know it)".[4] Since primitive propositions do not change their truth value based on the agent's epistemic state, this subtlety is irrelevant for propositional knowledge and knowability. For the purposes of this paper, we opt for the simplified "knowability" gloss of the $\square$ modality, and leave further investigation of this subtlety to future work.

3   Stalnaker's System Revised

Like Stalnaker, we augment a basic logic of knowledge with some additional axiom schemes that speak to the relationship between belief and knowledge. Unlike Stalnaker, however, we work with the language $\mathcal{L}_{K,\square,B}$ and take as our "basic logic of knowledge" the system

$$\mathsf{EL}_{K,\square} \quad = \quad \mathsf{S5}_K + \mathsf{S4}_\square + (\mathrm{KI}),$$

where (KI) denotes the axiom scheme $K\varphi \rightarrow \square \varphi$. As noted in Section 2.1, the evidence-in-hand conception of knowledge captured by the semantics for K is based on the premise that evidence-in-hand is completely transparent to the agent. That is, the agent is aware that she has the evidence she does and of what it entails and does not entail. In this sense, the agent is fully introspective with regard to the evidence-in-hand, and as such, K naturally emerges as an $\mathsf{S5}$-type modality.

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[3]It is not hard to see that $\llbracket \square \varphi \rrbracket^U = int(\llbracket \varphi \rrbracket^U)$ as one might expect; however, since the closure of $\llbracket \varphi \rrbracket^U$ need not be a subset of U, we have $\llbracket \Diamond \varphi \rrbracket^U = cl(\llbracket \varphi \rrbracket^U) \cap U$.

[4]This reading suggests a strong link to *conditional belief modalities*, which are meant to capture an agent's revised beliefs about how the world was before learning the new information. More precisely, a conditional belief formula $B^\varphi \psi$ is read as "if the agent would learn φ, then she would come to believe that ψ was the case (before the learning)" [7, p. 14]. Borrowing this interpretation, we might say that $\square \varphi$ represents hypothetical, conditional knowledge of φ where the condition consists in having some piece of evidence V entailing φ as evidence-in-hand: "if the agent were to have V as evidence-in-hand, she would know φ was the case (before having had the evidence)".

The system $\mathsf{EL}_{K,\square}$ was defined by Bjorndahl [12] and shown to be exactly the logic of topological subset spaces.

Theorem 1 ([12])**.** $\mathsf{EL}_{K,\square}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\square}$ *with respect to the class of all topological subset spaces: for every* $\varphi \in \mathcal{L}_{K,\square}$, φ *is provable in* $\mathsf{EL}_{K,\square}$ *if and only if* φ *is valid in all topological subset models.*

We strengthen $\mathsf{EL}_{K,\square}$ with the additional axiom schemes given in Table 3. Let $\mathsf{SEL}_{K,\square,B}$ denote the

(K_B)	$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$\vdash B\varphi \to B\square\varphi$	Responsible belief
(wF)	$\vdash B\varphi \to \Diamond\varphi$	Weak factivity
(CB)	$\vdash B(\square\varphi \vee \square\neg\square\varphi)$	Confident belief

Table 3: Additional axioms schemes for $\mathsf{SEL}_{K,\square,B}$

resulting logic. Schemes (sPI) and (KB) occur here just as they do in Stalnaker's original system (Table 2), and though (K_B) is not an axiom of Stal, it is derivable in that system. The remaining axioms involve the $\square$ modality and thus cannot themselves be part of Stalnaker's system; however, if we forget the distinction between $\square$ and K (and between $\Diamond$ and $\hat{K}$), all of them do hold in Stal, as made precise in Proposition 2.

Proposition 2. *Let* $t\colon \mathcal{L}_{K,\square,B} \to \mathcal{L}_{K,B}$ *be the map that replaces each instance of* $\square$ *with* K. *Then for every* φ *that is an instance of an axiom scheme from Table 3, we have* $\vdash_{\mathsf{Stal}} t(\varphi)$.

Proof. This is trivial for (sPI), (KB), and (RB). The scheme (K_B) follows immediately from the fact that Stal validates $\mathsf{KD45}_B$. After applying t, (wF) becomes $B\varphi \to \hat{K}\varphi$, which follows easily from the fact that $\vdash_{\mathsf{Stal}} B\varphi \leftrightarrow \hat{K}K\varphi$. Finally, under t, (CB) becomes $B(K\varphi \vee K\neg K\varphi)$, which follows directly from the aforementioned equivalence together with the fact that $\vdash_{\mathsf{S4}_K} \hat{K}K(K\varphi \vee K\neg K\varphi)$. $\qquad\square$

Thus, modulo the distinction between knowledge and knowablity, we make no assumptions beyond what follows from Stalnaker's own stipulations. Of course, the distinction between knowledge and knowability is crucial for us. Responsible belief differs from full belief in that K is replaced by $\square$, exactly as motivated in Section 2.1; it says that if you are sure of φ, then you must also be sure that there is some evidence that entails φ. Weak factivity and confident belief do not directly correspond to any of Stalnaker's axioms, but they are necessary to establish an analogue of Stalnaker's reduction of belief to knowledge (Proposition 3). Of course, one need not adopt these principles; indeed, rejecting them allows one to assent to the spirit of Stalnaker's premises without committing oneself to his conclusion that belief can be defined out of knowledge (or knowability). We return in Section 4 to consider weaker logics that omit one or both of (wF) and (CB).

Weak factivity can be understood, given (KI), as a strengthening of the formula $B\varphi \to \hat{K}\varphi$ (which is provable in Stal). Intuitively, it says that if you are certain of φ, then φ must be compatible with all the available evidence (in hand or not). Thus, while belief is not required to be factive—you can believe false things—(wF) does impose a weaker kind of connection to the world—you cannot believe *knowably* false things.

Confident belief expresses a kind of faith in the justificatory power of evidence. Consider the disjunction $\Box\varphi \vee \Box\neg\Box\varphi$, which says that φ is either knowable or, if not, that you could come to know that it is unknowable. This is equivalent to the negative introspection axiom for the $\Box$ modality, and does not hold in general; topologically speaking, it fails at the boundary points of the extension of $\Box\varphi$—where no measurement can entail φ yet every measurement leaves open the possibility that some further measurement will. What (CB) stipulates is that the agent is sure that they are not in such a "boundary case"—that every formula φ is either knowable or, if not, knowably unknowable.

Stalnaker's reduction of belief to knowledge has an analogue in this setting: every formula in $\mathcal{L}_{K,\Box,B}$ is provably equivalent to a formula in $\mathcal{L}_{K,\Box}$ via the following equivalence.

Proposition 3. *The formula* $B\varphi \leftrightarrow K\Diamond\Box\varphi$ *is provable in* $\mathsf{SEL}_{K,\Box,B}$.

Proof. We present an abridged derivation:

1.	$B\varphi \rightarrow \Diamond\Box\varphi$	(RB), (wF)
2.	$KB\varphi \rightarrow K\Diamond\Box\varphi$	(Nec$_K$), (K$_K$)
3.	$B\varphi \rightarrow KB\varphi$	(sPI)
4.	$B\varphi \rightarrow K\Diamond\Box\varphi$	CPL: 2, 3
5.	$B(\Box\varphi \vee \Box\neg\Box\varphi)$	(CB)
6.	$(\Box\varphi \vee \Box\neg\Box\varphi) \rightarrow (\Diamond\Box\varphi \rightarrow \varphi)$	(T$_\Box$), CPL
7.	$B(\Box\varphi \vee \Box\neg\Box\varphi) \rightarrow B(\Diamond\Box\varphi \rightarrow \varphi)$	(Nec$_K$), (KB), (K$_B$)
8.	$B(\Diamond\Box\varphi \rightarrow \varphi)$	CPL: 5, 7
9.	$B\Diamond\Box\varphi \rightarrow B\varphi$	(K$_B$)
10.	$K\Diamond\Box\varphi \rightarrow B\Diamond\Box\varphi$	(KB)
11.	$K\Diamond\Box\varphi \rightarrow B\varphi$	CPL: 9, 10
12.	$B\varphi \leftrightarrow K\Diamond\Box\varphi$	CPL: 4, 11. $\Box$

Thus, rather than being identified with the "epistemic possibility of knowledge" [33] as in Stalnaker's framework, to believe φ in this framework is to know that the knowability of φ is unfalsifiable. This is a bit of a mouthful, so consider for a moment the meaning of the subformula $\Diamond\Box\varphi$: in the informal language of evidence, this says that every piece of evidence is compatible not only with the truth of φ, but with the knowability of φ. In other words: no possible measurement can rule out the prospect that some further measurement will definitively establish φ. To believe φ, according to Proposition 3, is to know this.

This equivalence also tells us exactly how to extend topological subset space semantics to the language $\mathcal{L}_{K,\Box,B}$:

$$\begin{aligned}
(\mathcal{X},x,U) \models B\varphi \quad &\text{iff} \quad (\mathcal{X},x,U) \models K\Diamond\Box\varphi \\
&\text{iff} \quad (\forall y \in U)(y \in cl(int([\![\varphi]\!]^U))) \\
&\text{iff} \quad U \subseteq cl(int([\![\varphi]\!]^U)).
\end{aligned}$$

Thus, the agent believes φ at (x,U) just in case the interior of $[\![\varphi]\!]^U$ is dense in U. The collection of sets that have dense interiors on U forms a filter,[5] making it a good mathematical notion of *largeness*: sets with dense interior can be thought of as taking up "most" of the space. This provides an appealing intuition for the semantics of belief that runs parallel to that for knowledge: the agent *knows* φ at (x,U) iff φ is true at *all* points in U, whereas the agent *believes* φ at (x,U) iff φ is true at *most* points in U.

---

[5] A nonempty collection of subsets forms a filter if it is closed under taking supersets and finite intersections.

As mentioned in the introduction, this interpretation of belief as "truth at most points" (in a given domain) was first studied by Baltag et al. as a *topologically natural, evidence-based* notion of belief [4]. Though their motivation and conceptual underpinning differ from ours, the semantics for belief we have derived in this section essentially coincide with those given in [4]. We discuss this connection further in Section 5.

3.1 Technical results

Let (EQ) denote the scheme $B\varphi \leftrightarrow K\Diamond\Box\varphi$. It turns out that this equivalence is not only provable in $\mathsf{SEL}_{K,\Box,B}$, but in fact it characterizes $\mathsf{SEL}_{K,\Box,B}$ as an extension of $\mathsf{EL}_{K,\Box}$. To make this precise, let

$$\mathsf{EL}_{K,\Box}^+ = \mathsf{EL}_{K,\Box} + (\text{EQ}).$$

We then have:

Proposition 4. $\mathsf{EL}_{K,\Box}^+$ *and* $\mathsf{SEL}_{K,\Box,B}$ *prove the same theorems.*

From this it is not hard to establish soundness and completeness of $\mathsf{SEL}_{K,\Box,B}$:

Theorem 5. $\mathsf{SEL}_{K,\Box,B}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of topological subset models: for every* $\varphi \in \mathcal{L}_{K,\Box,B}$, φ *is valid in all topological subset models if and only if* φ *is provable in* $\mathsf{SEL}_{K,\Box,B}$.

Much work in belief representation takes the logical principles of $\mathsf{KD45}_B$ for granted (see, e.g., [24, 6, 15]). An important feature of $\mathsf{SEL}_{K,\Box,B}$ is that it *derives* these principles:

Proposition 6. *For every* $\varphi \in \mathcal{L}_B$, *if* $\vdash_{\mathsf{KD45}_B} \varphi$, *then* $\vdash_{\mathsf{SEL}_{K,\Box,B}} \varphi$.

In fact, $\mathsf{KD45}_B$ is not merely derivable in our logic—it completely characterizes belief as interpreted in topological models. That is, $\mathsf{KD45}_B$ proves exactly the validities expressible in the language $\mathcal{L}_B$:

Theorem 7. $\mathsf{KD45}_B$ *is a sound and complete axiomatization of* $\mathcal{L}_B$ *with respect to the class of all topological subset spaces: for every* $\varphi \in \mathcal{L}_B$, φ *is provable in* $\mathsf{KD45}_B$ *if and only if* φ *is valid in all topological subset models.*

Soundness follows easily from the above. The proof of completeness is more involved; it can be found in the full version of this paper.

4 Weaker Notions of Belief

In Section 3, we motivated the axioms of our system $\mathsf{SEL}_{K,\Box,B}$ in part by the fact that they allowed us to achieve a reduction of belief to knowledge-and-knowability in the spirit of Stalnaker's result. $\mathsf{SEL}_{K,\Box,B}$ includes several of Stalnaker original axioms (or modifications thereof), but also two new schemes: weak factivity (wF) and confident belief (CB). As noted, if we forget the distinction between knowledge and knowability, each of these schemes holds in Stal (Proposition 2). Nonetheless, in our tri-modal logic these two principles do not follow from the others: one can adopt (our translations of) Stalnaker's original principles while rejecting one or both of (wF) and (CB). In particular, this allows one to essentially accept all of Stalnaker's premises without being forced to the conclusion that belief is reducible to knowledge (or even knowledge-and-knowability). We are therefore motivated to generalize our earlier semantics in order to study weaker logics in which the belief modality is *not* definable and so requires its own semantic machinery.

In this section we do just this: we augment $\mathsf{EL}_{K,\square}$ with the axiom schemes given in Table 4 to form the logic $\mathsf{EL}_{K,\square,B}$, and prove that this system is sound and complete with respect to the new semantics defined below. We then consider logics intermediate in strength between $\mathsf{EL}_{K,\square,B}$ and $\mathsf{SEL}_{K,\square,B}$—specifically, those obtained by augmenting $\mathsf{EL}_{K,\square,B}$ with the axioms (D$_B$) (consistency of belief), (wF), or (CB)—and establish soundness and completeness results for these logics as well. As before, we rely on topological

(K$_B$)	$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$\vdash B\varphi \to B\square\varphi$	Responsible belief

Table 4: Additional axiom schemes for $\mathsf{EL}_{K,\square,B}$

subset models $\mathfrak{X} = (X, \mathfrak{T}, v)$ for the requisite semantic structure (see Section 2.2); however, we define the evaluation of formulas with respect to *epistemic-doxastic (e-d) scenarios*, which are tuples of the form (x, U, V) where (x, U) is an epistemic scenario, $V \in \mathfrak{T}$, and $V \subseteq U$. We call V the *doxastic range*.[6]

The semantic evaluation for the primitive propositions and the Boolean connectives is defined as usual; for the modal operators, we make use of the following semantic clauses:

$$
\begin{aligned}
(\mathfrak{X}, x, U, V) &\models K\varphi &\text{iff} &\quad U = \llbracket \varphi \rrbracket^{U,V} \\
(\mathfrak{X}, x, U, V) &\models \square\varphi &\text{iff} &\quad x \in int(\llbracket \varphi \rrbracket^{U,V}) \\
(\mathfrak{X}, x, U, V) &\models B\varphi &\text{iff} &\quad V \subseteq \llbracket \varphi \rrbracket^{U,V},
\end{aligned}
$$

where $\llbracket \varphi \rrbracket^{U,V} = \{x \in U : (\mathfrak{X}, x, U, V) \models \varphi\}$.

Thus, the modalities K and $\square$ are interpreted essentially as they were before, while the modality B is rendered as universal quantification over the doxastic range. Intuitively, we might think of V as the agent's "conjecture" about the way the world is, typically stronger than what is guaranteed by her evidence-in-hand U. On this view, states in V might be conceptualized as "more plausible" than states in $U \setminus V$ from the agent's perspective, with belief being interpreted as truth in all these more plausible states (see, e.g., [21, 10, 7, 11, 18] for more details on plausibility models for belief). Note that we do not require that $x \in V$; this corresponds to the intuition that the agent may have false beliefs. Note also that none of the modalities alter either the epistemic or the doxastic range; this is essentially what guarantees the validity of the strong introspection axioms.[7]

In order to distinguish these semantics from those previous, we refer to them as *epistemic-doxastic (e-d) semantics* for topological subset spaces.

Theorem 8. $\mathsf{EL}_{K,\square,B}$ *is a sound and complete axiomatization of $\mathcal{L}_{K,\square,B}$ with respect to the class of all topological subset spaces under e-d semantics.*

Call an e-d scenario (x, U, V) *consistent* if $V \neq \emptyset$, and call it *dense* if V is dense in U (i.e., if $U \subseteq cl(V)$).

---

[6]If we want to insist on *consistent* beliefs, we should add the axiom (D$_B$): $B\varphi \to \hat{B}\varphi$ (or, equivalently, $\hat{B}\top$) and require that $V \neq \emptyset$. We begin with the more general case, without these assumptions.

[7]We could, of course, consider even more general semantics that do not validate these axioms, but as our goal here is to understand the role of weak factivity and confident belief in the context of Stalnaker's reduction of belief to knowledge, we leave such investigations to future work.

Theorem 9. $\mathsf{EL}_{K,\Box,B} + (\mathrm{D}_B)$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios.* $\mathsf{EL}_{K,\Box,B} + (\mathrm{wF})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

4.1 Confident belief

It turns out that the strong semantics for the belief modality presented in Section 3, namely

$$(\mathcal{X}, x, U) \models B\varphi \quad \text{iff} \quad U \subseteq cl(int(\llbracket \varphi \rrbracket^U)),$$

does *not* arise as a special case of our new e-d semantics: there is no condition (e.g., density) one can put on the doxastic range V so that these two interpretations of $B\varphi$ agree in general. Roughly speaking, this is because the formulas of the form $\Box\varphi \vee \Box\neg\Box\varphi$ correspond to the open and dense sets, but in general one cannot find a (nonempty) open set V that is simultaneously contained in every open, dense set. As such, one cannot hope to validate (CB) in the e-d semantics presented above without also validating $B\bot$.

However, we can validate (CB) on topological subset spaces by altering the semantic interpretation of the belief modality so that, intuitively, it "ignores" *nowhere dense sets*.[8] Loosely speaking, this works because nowhere dense sets are exactly the complements of sets with dense interiors.

More precisely, we work with the same notion of e-d scenarios as before, but use the following semantics clauses:

$$
\begin{aligned}
(\mathcal{X}, x, U, V) &\models p && \text{iff} && x \in v(p) \\
(\mathcal{X}, x, U, V) &\models \neg\varphi && \text{iff} && (\mathcal{X}, x, U, V) \not\models \varphi \\
(\mathcal{X}, x, U, V) &\models \varphi \wedge \psi && \text{iff} && (\mathcal{X}, x, U, V) \models \varphi \text{ and } (\mathcal{X}, x, U, V) \models \psi \\
(\mathcal{X}, x, U, V) &\models K\varphi && \text{iff} && U = \llbracket \varphi \rrbracket^{U,V} \\
(\mathcal{X}, x, U, V) &\models \Box\varphi && \text{iff} && x \in int(\llbracket \varphi \rrbracket^{U,V}) \\
(\mathcal{X}, x, U, V) &\models B\varphi && \text{iff} && V \subseteq^* \llbracket \varphi \rrbracket^{U,V},
\end{aligned}
$$

where $\llbracket \varphi \rrbracket^{U,V} = \{x \in U : (\mathcal{X}, x, U, V) \models \varphi\}$, and we write $A \subseteq^* B$ iff $A \setminus B$ is nowhere dense. In other words, we interpret everything as before with the exception of the belief modality, which now effectively quantifies over *almost all* worlds in the doxastic range V rather than over all worlds.[9]

Theorem 10. $\mathsf{EL}_{K,\Box,B} + (\mathrm{CB})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics using the semantics given above: for all formulas* $\varphi \in \mathcal{L}_{K,\Box,B}$, *if* $\models \varphi$, *then* $\vdash_{\mathsf{EL}_{K,\Box,B}+(\mathrm{CB})} \varphi$. *Moreover,* $\mathsf{SEL}_{K,\Box,B}$ *is sound and complete with respect to these semantics for e-d scenarios where* $V = U$.

5 Conclusion and Discussion

When we think of knowledge as what is entailed by the "available evidence", a tension between two foundational principles proposed by Stalnaker emerges. First, that whatever the available evidence entails is believed ($K\varphi \to B\varphi$), and second, that what is believed is believed to be entailed by the available evidence ($B\varphi \to BK\varphi$). In the former case, it is natural to interpret "available" as, roughly speaking,

---

[8]A subset S of a topological space is called *nowhere dense* if its closure has empty interior: $int(cl(S)) = \emptyset$.

[9]Given a subset A of a topological space X, we say that a property P holds for *almost all* points in A just in case $A \subseteq^* \{x : P(x)\}$.

"currently in hand", whereas in the latter, intuition better accords with a broader interpretation of availability as referring to any evidence one could potentially access.

Being careful about this distinction leads to a natural division between what we might call "knowledge" and "knowability"; the space of logical relationships between knowledge, knowability, and belief turns out to be subtle and interesting. We have examined several logics meant to capture some of these relationstips, making essential use of *topological* structure, which is ideally suited to the representation of evidence and the epistemic/doxastic attitudes it informs. In this refined setting, belief can also be defined in terms of knowledge and knowability, provided we take on two additional principles, "weak factivity" (wF) and "confident belief" (CB); in this case, the semantics for belief have a particularly appealing topological character: roughly speaking, a proposition is believed just in case it is true in *most* possible alternatives, where "most" is interpreted topologically as "everywhere except on a nowhere dense set".

This interpretation of belief first appeared in the topological belief semantics presented in [4]: Baltag et al. take the believed propositions to be the sets with *dense interiors* in a given *evidential* topology. Interestingly, however, although these semantics essentially coincide with those we present in Section 3, the motivations and intuitions behind the two proposals are quite different. Baltag et al. start with a subbase model in which the (subbasic) open sets represent pieces of evidence that the agent has *obtained directly* via some observation or measurement. They do not distinguish between evidence-in-hand and evidence-out-there as we do; moreover, the notion of belief they seek to capture is that of *justified belief*, where "justification", roughly speaking, involves having evidence that cannot be defeated by any other available evidence. (They also consider a weaker, defeasible type of knowledge, *correctly justified belief*, and obtain topological semantics for it under which Stalnaker's original system Stal is sound and complete.) The fact that two rather different conceptions of belief correspond to essentially the same topological interpretation is, we feel, quite striking, and deserves a closer look.

Despite the elegance of this topological characterization of belief, our investigation of the interplay between knowledge, knowability, and belief naturally leads to consideration of weaker logics in which belief is not interpreted in this way. In particular, we focus on the principles (wF) and (CB) and what is lost by their omission. Again we rely on topological subset models to interpret these logics, proposing novel semantic machinery to do so. This machinery includes the introduction of the *doxastic range* and, perhaps more dramatically, a modification to the semantic satisfaction relation ($\approx$) that builds the topological notion of "almost everywhere" quantification directly into the foundations of the semantics. We believe this approach is an interesting area for future research, and in this regard our soundness and completeness results may be taken as proof-of-concept.

Acknowledgements

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